Axiomatic characterizations of some simple risk-sharing rules

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Risk-sharing arrangements

- Risk-sharing arrangements involve multiple participants agreeing to share the costs or losses associated with uncertain events across themselves.
  - Agreements could be formal or informal.
  - Participants can be individuals, communities, businesses, organizations, or governments.
- There are several reasons for getting into such arrangements:
  - They help reduce the financial impact of adverse events on any single person or entity.
  - Because risk is spread, there is a more efficient use of resources.
  - Risk-sharing encourages a stronger, more resilient communities and partnerships.
  - Such arrangements enable participants to access more resources, opportunities, or markets otherwise considered too risky.
  - They make investments of large-scale projects, such as infrastructure that may involve significant risks, become possible.

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Examples of risk-sharing arrangements

- Centralized insurance
  - Traditional insurance and government-run programs, with decisions made by a centralized authority and regulated by government bodies.
- Decentralized insurance
  - Modern insurance model without the reliance of a traditional insurer or intermediary.
  - Mutual aid societies, Takaful (Islamic insurance), Peer-to-peer (P2P) insurance, DAO-governed insurance (decentralized autonomous organizations), Smart contract-based insurance.
- Parametric insurance, coverage that pays out a predetermined amount based on a specific trigger, can be centralized or decentralized.
- The growth of decentralized insurance schemes has heightened the importance for defining risk-sharing rules.

Preliminaries

- All random variables (r.v.'s) are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Let χ be an appropriate set of r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$.
  - Interpret χ as the collection of losses (risks) of interest.
  - χ is assumed to be a convex cone of r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$ so that it is closed under positive scalar multiplication and under addition.
  - Assume that 0 is in χ.
  - Depending on the situation at hand, χ could be L^1 or L^2 or L^∞, or the set of all r.v.'s, denoted as L^0.
  - Also, for any L^p, the subset L^p_+ consisting of all non-negative elements of L^p might be a suitable choice.
  - We will always silently assume that χ only contains non-negative r.v.'s (losses), although several results we present hereafter remain to hold (or can easily be adapted) when this non-negativity restriction is not made.

Reallocation

- Each economic agent, labeled $i = 1, 2, \dots, n$, faces a random loss X_i at time 0, and would incur a loss equal to the realization of X_i at time 1.
- $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ is the loss vector with cdf $F_{\boldsymbol{X}}$ and marginals $F_{X_1}, F_{X_2}, \dots, F_{X_n}$, respectively.
- The total or aggregate loss experienced by the n agents is

$$S_{\boldsymbol{X}} = \sum_{i=1}^{n} X_i.$$

- For any pool $\boldsymbol{X} \in \chi^n$ with aggregate loss $S_{\boldsymbol{X}}$, the set $\mathcal{A}_{\boldsymbol{X}}$, a **reallocation** of $\boldsymbol{X}$, is defined as

$$\mathcal{A}_{\boldsymbol{X}} = \left\{ (Y_1, Y_2, \dots, Y_n) \in (L^0)^n \,\middle|\, \sum_{i=1}^{n} Y_i = S_{\boldsymbol{X}} \right\}.$$

Risk-sharing

- Risk-sharing (RS) within a pool $\boldsymbol{X} \in \chi^n$ is a two-stage process:
  - *ex-ante step* (Time 0): X_i within the pool are reallocated by transforming $\boldsymbol{X}$ into another random vector $\boldsymbol{C}[\boldsymbol{X}] \in \mathcal{A}_{\boldsymbol{X}}$:

$$\boldsymbol{C}[\boldsymbol{X}] = \left(C_1[\boldsymbol{X}], C_2[\boldsymbol{X}], \ldots, C_n[\boldsymbol{X}]\right).$$

  - *ex-post step* (Time 1): each participant receives the realization of X_i from the pool and pays the realization of $C_i[\boldsymbol{X}]$ to the pool.
- $C_i[\boldsymbol{X}]$ is referred to as the contribution of participant i to the pool, while $\boldsymbol{C}[\boldsymbol{X}]$ is called the contribution vector.
- The risk-sharing process must satisfy the **full allocation condition**:

$$\sum_{i=1}^{n} C_i[\boldsymbol{X}] = \sum_{i=1}^{n} X_i.$$

- One perspective for this condition is by re-writing:

$$\sum_{i=1}^{n} (X_i - C_i[\boldsymbol{X}]) = 0.$$

  Some will lose, but some will gain.

Risk-sharing (RS) rule

- A risk-sharing rule for a given group of n participants, each with losses in χ, is a mapping $\boldsymbol{C}$ that transforms any pool $\boldsymbol{X}$ in χ^n into a contribution vector $\boldsymbol{C}[\boldsymbol{X}]$ in $\mathcal{A}_{\boldsymbol{X}}$:

$$\boldsymbol{X} \in \chi^n \rightarrow \boldsymbol{C}[\boldsymbol{X}] \in \mathcal{A}_{\boldsymbol{X}}.$$

- $\boldsymbol{C}$ is a more general function from χ^n to (a subset of) $(L^0)^n$, not just from $\mathbb{R}^n$ to (a subset of) $\mathbb{R}^n$.
- At time 0, $\boldsymbol{C}[\boldsymbol{X}]$ is a random vector, depending on $\boldsymbol{X}$ and potentially on other sources of randomness.
  - Thus, $\boldsymbol{C}[\boldsymbol{X}]$ is not necessarily measurable with respect to $\sigma(\boldsymbol{X})$.
- The RS rule is established and defined at time 0, before any losses occur.
- It establishes how the aggregate loss, observable at time 1, will be shared among participants.
  - Designing and implementing the suitable RS rule is crucial for ensuring success and sustainability of the RS arrangement.

Some examples of RS rules

- Self-insurance RS rule:

$$\boldsymbol{C}^{\mathrm{si}}[\boldsymbol{X}] = (X_1, X_2, \ldots, X_n).$$

- Order statistics RS rule:

$$\boldsymbol{C}^{\mathrm{ord}}[\boldsymbol{X}] = \left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right).$$

- All-in-one RS rule:

$$\boldsymbol{C}^{\mathrm{all}}[\boldsymbol{X}] = (S_{\boldsymbol{X}}, 0, \ldots, 0).$$

- Mean-proportional RS rule:

$$\boldsymbol{C}^{\mathrm{mean}}[\boldsymbol{X}] = \left(\frac{\mathbb{E}[X_1]}{\mathbb{E}[S_{\boldsymbol{X}}]} S_{\boldsymbol{X}}, \frac{\mathbb{E}[X_2]}{\mathbb{E}[S_{\boldsymbol{X}}]} S_{\boldsymbol{X}}, \ldots, \frac{\mathbb{E}[X_n]}{\mathbb{E}[S_{\boldsymbol{X}}]} S_{\boldsymbol{X}}\right).$$

- Conditional mean RS rule:

$$\boldsymbol{C}^{\mathrm{cmrs}}[\boldsymbol{X}] = (\mathbb{E}[X_1|S_{\boldsymbol{X}}], \mathbb{E}[X_2|S_{\boldsymbol{X}}], \ldots, \mathbb{E}[X_n|S_{\boldsymbol{X}}]).$$

Existing literature

- Risk-sharing has roots in actuarial science and insurance:
  - Concept dates back to ancient Babylon, around 1750 BCE, in the *Code of Hammurabi*.
- Denuit, Dhaene, and Robert (2022) explore properties of several risk-sharing methods including CMRS, introduced in Denuit and Dhaene (2012).
- Denuit and Robert (2021) analyze risk sharing rules, including linear risk-sharing rules, for large pools with heterogeneous losses, including P2P setups.
- Levantasi and Piscopo (2022) and Clemente, Levantasi, and Piscopo (2023) examine P2P insurance with safety margins for fluctuating total losses.
- Dhaene, et al. (2024) introduce the quantile RS rule.
- Feng, Liu, and Taylor (2023) propose a P2P model using convex programming for fair risk-sharing, with focus to flood risk pooling.
- Ghossoub, Zhu, and Chong (2024) investigate RS allocations that account for the risk appetite of each participant with respect to tail events.

Axiomatic foundations

- **Axiomatic foundations** of decision-making rules like insurance pricing have been widely explored, e.g., Wang, Young, and Panjer (1997), but limited on RS rules.
- Denuit, Dhaene, and Robert (2022) consider an extensive list of properties that RS rules might obey.
- Jiao, et al. (2022) provide an axiomatic characterization of the CMRS rule.
- Dhaene, et al. (2024) present an axiomatic characterization of the quantile RS rule.
- Our work expands on this by developing axiomatic frameworks for some simple RS rules.
    - Constructing RS rules using an axiomatic approach is just one approach; another is solving an optimization problem as in Yang and Wei (2024).

RS rules characterized in our paper

- Uniform RS rule:

$$\boldsymbol{C}^{\text{uni}}[\boldsymbol{X}] = \left(\frac{S_{\boldsymbol{X}}}{n}, \frac{S_{\boldsymbol{X}}}{n}, \dots, \frac{S_{\boldsymbol{X}}}{n} \right).$$

- Class of q-proportional RS rule:

$$\boldsymbol{C}^{\text{prop}}[\boldsymbol{X}] = \left(\frac{q[X_1]}{\sum_{k=1}^{n} q[X_k]} S_{\boldsymbol{X}}, \frac{q[X_2]}{\sum_{k=1}^{n} q[X_k]} S_{\boldsymbol{X}}, \dots, \frac{q[X_n]}{\sum_{k=1}^{n} q[X_k]} S_{\boldsymbol{X}} \right),$$

for risk metric $q : \chi \to \mathbb{R}_0^+$, and for any pool $\boldsymbol{X}$ with at least one $q[X_j] > 0$.

- Class of (q_1, q_2)-based linear RS rule:

$$\begin{aligned} \boldsymbol{C}^{\text{lin}}[\boldsymbol{X}] &= \Bigg(q_1[X_1] + \frac{q_2[X_1, S_{\boldsymbol{X}}]}{\sum_{k=1}^{n} q_2[X_k, S_{\boldsymbol{X}}]} (S_{\boldsymbol{X}} - \sum_{k=1}^{n} q_1[X_k]), \\ &\quad \dots, q_1[X_n] + \frac{q_2[X_n, S_{\boldsymbol{X}}]}{\sum_{k=1}^{n} q_2[X_k, S_{\boldsymbol{X}}]} (S_{\boldsymbol{X}} - \sum_{k=1}^{n} q_1[X_k]) \Bigg), \end{aligned}$$

for risk metrics $q_1 : \chi \to \mathbb{R}_0^+$ and $q_1 : \chi^2 \to \mathbb{R}_0^+$, and for any pool $\boldsymbol{X}$ where denominator is not zero.

Examples of q-proportional RS rules

- Mean-proportional RS rule:

$$C_i^{\text{mean}}[\boldsymbol{X}] = \frac{\mathbb{E}[X_i]}{\sum_{k=1}^{n} \mathbb{E}[X_k]} S_{\boldsymbol{X}}$$

- Variance-proportional RS rule:

$$C_i[\boldsymbol{X}] = \frac{\mathrm{var}(X_i)}{\sum_{k=1}^{n} \mathrm{var}(X_k)} S_{\boldsymbol{X}}$$

  A variation to this is using standard deviation.

- Scenario-based proportional RS rule:

$$C_i^{\text{scen,prop}}[\boldsymbol{X}] = \frac{X_i(\omega^*)}{\sum_{k=1}^{n} X_k(\omega^*)} S_{\boldsymbol{X}},$$

  where ω^* represents a state of the world.

Illustration: scenario-based proportional RS

Consider a pool of 4 participants who agreed to share damages incurred after a hurricane. The group agreed to share damages proportionately according to the preset scenario ω^* measured by wind scale of at least 3:

wind scale ω	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$X_4(\omega)$	$S_{\mathbf{X}}(\omega)$
1	1	1	1	1	4
2	2	4	5	9	20
≥ 3 ω^*	5	9	15	31	60

- The hurricane resulted in a wind scale of 2.
- The damage vector is $(2, 4, 5, 9)$.
- The total damage is 20.
- The contribution vector will be $(1.67, 3.00, 5.00, 10.33)$.

Examples of (q_1, q_2)-proportional RS rules

- Covariance-based linear RS rule: $q_1[X_i] = \mathbb{E}[X_i]$, $q_2[X_i, S] = \text{cov}[X_i, S]$

$$C_i^{\text{cov}}[\boldsymbol{X}] = \mathbb{E}[X_i] + \frac{\text{cov}(X_i, S_{\boldsymbol{X}})}{\text{var}(S_{\boldsymbol{X}})}(S_{\boldsymbol{X}} - \mathbb{E}[S_{\boldsymbol{X}}])$$

- Variance-based linear RS rule: $q_1[X_i] = \mathbb{E}[X_i]$, $q_2[X_i, S] = \text{var}(X_i)$

$$C_i^{\text{var}}[\boldsymbol{X}] = \mathbb{E}[X_i] + \frac{\text{var}(X_i)}{\sum_{k=1}^{n} \text{var}(X_k)}(S_{\boldsymbol{X}} - \mathbb{E}[S_{\boldsymbol{X}}])$$

- Scenario-based linear RS rule: $q_1[X_i] = X_i(\omega^*)$,
$q_2[X_i, S] = (X_i(\overline{\omega}) - X_i(\underline{\omega}))(S(\overline{\omega}) - S(\underline{\omega}))$

$$C_i^{\text{scen,lin}}[\boldsymbol{X}] = X_i(\omega^*) + \frac{X_i(\overline{\omega}) - X_i(\underline{\omega})}{S_{\boldsymbol{X}}(\overline{\omega}) - S_{\boldsymbol{X}}(\underline{\omega})}(S_{\boldsymbol{X}} - S_{\boldsymbol{X}}(\omega^*)),$$

where ω^*, $\overline{\omega}$, and $\underline{\omega}$ represents three states of the world.

Reshuffling

- A reshuffle of pool $\boldsymbol{X}$ is a random vector $\boldsymbol{X}^{\pi}$ defined by

$$\boldsymbol{X}^{\pi} = (X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}),$$

  where $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ represents a permutation of $\{1, \ldots, n\}$.

- Reshuffling property: A RS rule $\boldsymbol{C}$ satisfies the reshuffling property if for any $\boldsymbol{X} \in \chi^n$ and any of its reshuffles $\boldsymbol{X}^{\pi}$, the following holds:

$$C_i[\boldsymbol{X}^{\pi}] = C_{\pi(i)}[\boldsymbol{X}], \quad \text{for any } i = 1, \ldots, n.$$

- Consider the pool $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ and the reshuffle $\boldsymbol{X}^{\pi} = (X_3, X_4, X_1, X_2)$. According to the reshuffling property, contribution vector must be

$$\boldsymbol{C}[\boldsymbol{X}^{\pi}] = (C_3[\boldsymbol{X}], C_4[\boldsymbol{X}], C_1[\boldsymbol{X}], C_2[\boldsymbol{X}]).$$

- Losses and contributions are interconnected such that when participants exchange their individual losses, their contributions are exchanged correspondingly.

Source-anonymous contributions

The contributions of a RS rule $\boldsymbol{C}$ are said to be source-anonymous if, for any pool $\boldsymbol{X}$ and any of its reshuffles $\boldsymbol{X}^{\pi}$, it holds that

$$C_i[\boldsymbol{X}^{\pi}] = C_i[\boldsymbol{X}] \quad \text{for any } i = 1, \ldots, n.$$

- Source-anonymity of a risk-sharing rule means that the contributions are not tied to who specifically incurs the losses $X_1, X_2, \ldots, X_n$.
- Consider the pool $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ and any reshuffle, say, $\boldsymbol{X}^{\pi} = (X_3, X_4, X_1, X_2)$. For source-anonymous contributions, contribution vector must be
$$\boldsymbol{C}[\boldsymbol{X}^{\pi}] = (C_1[\boldsymbol{X}], C_2[\boldsymbol{X}], C_3[\boldsymbol{X}], C_4[\boldsymbol{X}]).$$
- Contributions are determined by the individual losses, but the source of these individual losses is irrelevant for determining these contributions.
- The order statistics RS rule satisfies this property.

Aggregate contributions

A RS rule is said to have aggregate contributions if for any pool $\boldsymbol{X}$ there exists a function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^n$ such that the contributions of $\boldsymbol{X}$ are given by:

$$C_i[\boldsymbol{X}] = h_i(S_{\boldsymbol{X}}) \quad \text{for any } i = 1, \ldots, n.$$

- Randomness of the contributions is solely due to the randomness of the aggregate loss.
- Contributions $\boldsymbol{C}[\boldsymbol{X}]$ are measurable with respect to $\sigma(S_{\boldsymbol{X}})$.
- Only realization of $S_{\boldsymbol{X}}$ is revealed; no need to reveal realizations of X_i's. But, it may involve risk metrics of X_i's such as their means.
- Also referred to as '*non-olet*' property. See Borch (1960) and Feng (2023).

Strongly aggregate contributions

A RS rule is said to have strongly aggregate contributions if there exists a function $\mathbf{h} : \mathbb{R} \to \mathbb{R}^n$ such that contributions of any pool $\boldsymbol{X}$ are given by:

$$C_i[\boldsymbol{X}] = h_i(S_{\boldsymbol{X}}) \quad \text{for any } i = 1, \ldots, n.$$

- Contributions $\boldsymbol{C}[\boldsymbol{X}]$ are also measurable with respect to $\sigma(S_{\boldsymbol{X}})$.
- Indeed, strongly aggregate are also aggregate contributions, but not vice versa.
- The function h_i that tells us how to divide the aggregate loss across participants is the same across different risk pools.
- Uniform RS rule has strongly aggregate and hence, aggregate contributions.
- Mean-proportional RS rule has aggregate but not strongly aggregate contributions.

Risk-sharing rules satisfying the various properties

Table 1: Properties of some risk-sharing rules

Risk-sharing (RS) rules	Reshuffling	Source-anonymous contributions	Aggregate contributions	Strongly aggregate contributions
Order statistics RS	–	✓	–	–
Conditional mean RS	✓	–	✓	–
Mean-proportional RS	✓	–	✓	–
Scenario-based proportional RS	✓	–	✓	–
Scenario-based linear RS	✓	–	✓	–
All-in-one RS	–	✓	✓	✓
Uniform RS	✓	✓	✓	✓

Characterizing the uniform RS rule

Theorem 1: Risk-sharing rule $\boldsymbol{C}$ is the uniform RS rule if and only if it satisfies the following two axioms:

1: $\boldsymbol{C}$ satisfies the reshuffling property.

2: $\boldsymbol{C}$ has source-anonymous contributions.

- Axioms 1 and 2 are independent.

Theorem 2: Risk-sharing rule $\boldsymbol{C}$ is the uniform RS rule if and only if the following two axioms hold:

1: $\boldsymbol{C}$ satisfies the reshuffling property.

3: $\boldsymbol{C}$ has strongly aggregate contributions.

- Axioms 1 and 3 are independent.

Modifying the axioms for q-proportional RS rule

Source-anonymous contribution-over-q ratios The contribution-over-q ratios of a RS rule $\boldsymbol{C}$ are said to be source-anonymous if, for any pool $\boldsymbol{X}$ and any reshuffling $\boldsymbol{X}^{\pi}$, the following holds:

$$C_i\left[\boldsymbol{X}^{\pi}\right] = \frac{q\left[X_{\pi(i)}\right]}{q\left[X_i\right]} C_i\left[\boldsymbol{X}\right] \quad \text{for any } i = 1, \ldots, n \quad \text{with } q\left[X_i\right] > 0.$$

Strongly aggregate contribution-over-q ratios Consider normalized and additive risk metric $q : \chi \to \mathbb{R}_0^+$. A RS rule $\boldsymbol{C}$ in χ^n has strongly aggregate contribution-over-q ratios if there exists a function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^n$ such that for any $\boldsymbol{X} \in \chi^n$ with at least one $q\left[X_j\right] > 0$, the contributions are given by:

$$C_i\left[\boldsymbol{X}\right] = q\left[X_i\right] \times h_i\left(S_{\boldsymbol{X}}, q\left[S_{\boldsymbol{X}}\right]\right), \quad \text{for any } i = 1, \ldots, n,$$

- Risk metric $q : \chi \to \mathbb{R}_0^+$ is said to be normalized if $q[0] = 0$ and additive if $q\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} q[X_k]$, for any $\boldsymbol{X} \in \chi^n$.

Theorem 3: Consider the risk metric $q : \chi \to \mathbb{R}_0^+$. A RS rule $\boldsymbol{C}$ in χ^n is the q-proportional RS rule if and only if it satisfies the following two axioms:

1. $\boldsymbol{C}$ satisfies the reshuffling property.
4. $\boldsymbol{C}$ has source-anonymous contribution-over-q ratios.

- Axioms 1 and 4 are independent.

Theorem 4: Consider the normalized and additive risk metric $q : \chi \to \mathbb{R}_0^+$. A RS rule $\boldsymbol{C}$ in χ^n is the q-proportional RS rule if and only if it satisfies the following axiom:

5. $\boldsymbol{C}$ has strongly aggregate contribution-over-q ratios.

Weighted q-proportional RS rule

For the weighted q-proportional RS rule, the contribution of participant i is

$$C_i[\mathbf{X}] = \frac{w_i q[X_i]}{\sum_{k=1}^{n} w_k q[X_k]} S_{\mathbf{X}},$$

where $w_1, \ldots, w_n$ are positive real numbers.

- Contribution depends on the aggregate loss, but also on participants' identities (through w_j's) and their respective random losses (through $q[X_j]$'s).
- The w_j's adjust risk metrics, the $q[X_j]$'s, to reflect
  - Data quality which affects trustworthiness.
  - Adjustment factor for model uncertainty.
- If $w_1, \ldots, w_n$ are not all equal, then weighted q-proportional RS rule fails reshuffling axiom 1 and source-anonymous contribution-over-q ratios axiom 4, therefore, not a q-proportional RS rule.

Modifying the axioms for (q_1, q_2)-based linear RS rule

Consider the risk metrics $q_1 : \chi \to \mathbb{R}_0^+$ and $q_2 : \chi^2 \to \mathbb{R}$.

Source-anonymous (q_1, q_2)-standardized contributions The (q_1, q_2)-standardized contributions of a RS rule $\boldsymbol{C}$ are said to be source-anonymous if, for any pool $\boldsymbol{X}$ and any of its reshuffles $\boldsymbol{X}^\pi$, the following conditions hold:

$$C_i[\mathbf{X}^\pi] - q_1[X_{\pi(i)}] = \frac{q_2[X_{\pi(i)}, S_\mathbf{X}]}{q_2[X_i, S_\mathbf{X}]}(C_i[\mathbf{X}] - q_1[X_i]), \text{ for any } i = 1, \ldots, n,$$

with $q_2[X_i, S_\mathbf{X}] \neq 0$.

Strongly aggregate (q_1, q_2)-based standardized contributions A RS rule $\boldsymbol{C}$ has strongly aggregate (q_1, q_2)-based standardized contributions if there exists a function $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^n$ such that the relative contributions for any $\boldsymbol{X} \in \chi^n$ with at least one $q_2[X_j, S_\mathbf{X}] \neq 0$ for at least one $j \in \{1, \ldots, n\}$, can be expressed as:

$$C_i[\mathbf{X}] = q_1[X_i] + q_2[X_i, S_\mathbf{X}] \cdot h_i(S_\mathbf{X}, q_1[S_\mathbf{X}], q_2[S_\mathbf{X}, S_\mathbf{X}]) \quad \text{for any } i = 1, \ldots, n.$$

Characterizing the (q_1, q_2)-based linear RS

Consider the risk metrics $q_1 : \chi \to \mathbb{R}_0^+$ and $q_2 : \chi^2 \to \mathbb{R}$.

Theorem 5: A RS rule $\boldsymbol{C}$ is the (q_1, q_2)-based linear RS rule if and only if it satisfies the following two axioms:

1. $\boldsymbol{C}$ satisfies the reshuffling property.
6. $\boldsymbol{C}$ has source-anonymous (q_1, q_2)-standardized contributions.

- Axioms 1 and 6 considered in Theorem 5 are independent.

Theorem 6: Let $q_1[0] = 0$ and $q_2[0, \cdot] = 0$, and let both measures be additive in their first argument. Then a RS rule $\boldsymbol{C}$ is the (q_1, q_2)-based linear RS rule if and only if it satisfies the following axiom:

7. $\boldsymbol{C}$ has aggregated standardized contributions.

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The covariance-based linear RS rule

Consider the covariance-based linear RS rule where we choose

$$q_1[X_i] = \mathbb{E}[X_i] \qquad\qquad q_2[X_i, S_{\mathbf{X}}] = \mathrm{cov}[X_i, S_{\mathbf{X}}].$$

- It is easy to show that this RS rule has source-anonymous (q_1, q_2)-standardized contributions.
- It can also be shown that this has strongly aggregate (q_1, q_2)-based standardized contributions by writing the contribution formula as

$$\begin{aligned} C_i^{\mathrm{cov}}[\mathbf{X}] &= \mathbb{E}[X_i] + \mathrm{cov}(X_i, S_{\mathbf{X}}) \cdot \frac{S_{\mathbf{X}} - \mathbb{E}[S_{\mathbf{X}}]}{\mathrm{var}(S_{\mathbf{X}})} \\ &= \mathbb{E}[X_i] + \mathrm{cov}(X_i, S_{\mathbf{X}}) \cdot h_i(S_{\mathbf{X}}, \mathbb{E}[S_{\mathbf{X}}], \mathrm{cov}(S_{\mathbf{X}}, S_{\mathbf{X}})) \quad \text{for any } i = 1, \ldots, n. \end{aligned}$$

Concluding remarks

- This paper explores axiomatic characterizations of simple risk-sharing rules.
- We describe the uniform RS rule using three essential properties: (1) Reshuffling; (2) Source-anonymous contributions; and (3) Strongly aggregate contributions
- These elementary axioms form the basis for two other broader classes of RS rules:
  - the q-proportional RS rules and
  - the (q_1, q_2)-based linear RS rules.
- These axiomatic characterizations enable us to introduce new RS rules, such as the scenario-based RS rules, which provide for novel examples of the q-proportional RS rules and (q_1, q_2)-based linear RS rules.
  - Under these rules, risk-sharing adapts to predefined scenarios such as extreme or typical events, which allow for a more dynamic approach.
  - Innovation does not require probability knowledge, but relies instead on expert judgments or opinions.

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- Thank you -